Technical Comments

Comments on "Optimal Controls for Out-of-Plane Motion about the Translunar Libration Point"

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IN the last paragraph of the Synoptic for Heppenheimer's recent paper,[1] the following statement is made: "These results are in conflict with a period control proposed by Farquhar.[2] Farquhar's solution, however, involves substantially higher cost and in addition represents a single optimum and not a family of local optima. Moreover, his solution is objectionable on theoretical grounds since its characteristics are strongly dependent upon the small parameter e, the lunar eccentricity, and his solution actually fails to exist for $e = 0$. The preceding solution (i.e., Heppenheimer's solution), though computed for the case $e = 0$, requires only slight modification to accommodate the actual case of $e \ll 1$ (i.e., A_z must be increased). In this light, Farquhar's requirement that $e > 0$ appears artificial." A reply to Heppenheimer's criticism is given below.

Heppenheimer's remarks concerning the lunar eccentricity are unfounded since Ref. 2 used a model that neglected this quantity. Evidently Heppenheimer is referring to the small frequency difference $\epsilon \equiv \omega_y - \omega_z = 0.07647$. Clearly, there is nothing "artificial" about $\epsilon > 0$.

Furthermore, it was never stated that the "phase-jump control" of Ref. 2 is the most economical z-axis control. Actually, it had already been pointed out in a previous paper[3] that a continuous sinusoidal z-axis control is more economical than the phase-jump control. The primary motivation for introducing the phase-jump method in Ref. 2 was to demonstrate that it would be possible to guarantee a nonocculted trajectory even when the time interval between control pulses is as long as 3 months.

The fuel consumption estimates of Ref. 1 also deserve some comment. The average ΔV cost for the impulsive z-axis control discussed in Ref. 1† can be written as‡

$$\overline{\Delta V} = \epsilon\omega_z A_z \sin\psi_c/\psi_c \tag{1}$$

where $\epsilon = \omega_y - \omega_z = 0.07647$, $\omega_y = 1.86265$, $\omega_z = 1.78618$, $\psi_c = \epsilon\Delta t/2$, and Δt is the time interval between impulses. (For $\Delta t = 7.334$ days, $\psi_c = 0.06449$ in normalized units which corresponds to Mode 1 of Heppenheimer's paper.) The amplitude of the z oscillation, A_z, must be large enough to guarantee that the satellite trajectory in the yz plane never enters the occulted zone. To determine A_z, it is necessary to begin with the basic equations for the satellite trajectory in the yz plane:

$$y = A_y \cos\omega_y t, \quad z = A_z \sin(\omega_y t - \psi) \tag{2}$$

where $\psi \equiv \psi_0 + \epsilon t$ and ψ_0 is the initial phase angle. Since ϵ is small, the phase angle ψ will be approximated by an average value through one cycle. It follows from Eq. (2) that

$$2r^2 \equiv 2(y^2 + z^2) = (A_y^2 + A_z^2) + (A_y^2 - A_z^2 \cos 2\psi) \cos 2\omega_y t - (A_z^2 \sin 2\psi) \sin 2\omega_y t \tag{3}$$

For a trajectory that just touches the occulted zone,

$$2r_0^2 = (A_y^2 + A_z^2) - [(A_y^2 - A_z^2 \cos 2\psi_c)^2 + (A_z^2 \sin 2\psi_c)^2]^{1/2} \tag{4}$$

where r_0 is the radius of the occulted zone ($r_0 = 3100$ km) and ψ_c is the phase angle for the cycle that just misses the occulted zone. Therefore, the minimum value of A_z that is needed to ensure nonocculation is given by

$$A_z^2 = 2r_0^2/\{(1 + k^2) - [1 - 2k^2 \cos 2\psi_c + k^4]^{1/2}\} \tag{5}$$

where $A_y = kA_z$ and it is specified that $k \geq 1$. For $k = 1$ (i.e., $A_y = A_z$), Eq. (5) reduces to

$$A_z = r_0/(1 - \sin\psi_c)^{1/2} \tag{6}$$

In Heppenheimer's paper, the influence of A_y is not considered and the minimum value of A_z is taken as

$$A_z = r_0/\cos\psi_c \tag{7}$$

A comparison of the average ΔV costs for several different cases is given in Table 1. Notice that for $k = 1$ there is a 42.7% fuel penalty when the interval between thrusts is increased tenfold. Heppenheimer states that this penalty is only 16.5%. Finally it should be mentioned that all of the ΔV costs given in Table 1 are slightly optimistic since the effects of nonlinearities, lunar eccentricity, and the sun's gravitational field have been neglected.

Table 1 Average ΔV requirements for impulsive z-axis controls (r_0 = 3100 km)

	Revised analysis [Eqs. (1) and (5)]		Ref. 1 [Eqs. (1) and (7)]	
	A_z, km	$\overline{\Delta V}$, fps/yr	A_z, km	$\overline{\Delta V}$, fps/yr
Δt = 7.334 days (Mode 1 of Ref. 1)				
$k = 1$	3205	321	3106	311
$k = 1.5$	3112	312		
$k = 2$	3109	311		
Δt = 73.34 days (Mode 10 of Ref. 1)				
$k = 1$	4908	458	3879	362
$k = 1.5$	4267	398		
$k = 2$	4079	381		
Δt = 93.04 days				
$k = 1$	5964	533	4535	405
$k = 1.078$	5755	514		
$k = 1.5$	5135	459		
$k = 2$	4859	434		

[a] For Δt = 93.04 days and k = 1.078 the phase jump control of Ref. 2 gives $\overline{\Delta V}$ = 482 fps/yr.

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† It should be noted that this z-axis control technique was originally proposed in Ref. 4.

‡ The usual normalizations of the restricted 3-body problem are used here. That is, the following quantities are taken as unity: 1) sum of the masses of the Earth and moon; 2) mean Earth-moon distance (384,405 km); 3) mean angular rate of the moon around the Earth (0.22997 rad/day). The normalized value of $\overline{\Delta V}$ can be converted to fps/yr by multiplying by the factor 2.819637×10^6.

References

[1] Heppenheimer, T. A., "Optimal Controls for Out-of-Plane Motion about the Translunar Libration Point," *Journal of Spacecraft and Rockets*, Vol. 7, No. 9, Sept. 1970, pp. 1087–1092.

[2] Farquhar, R. W., "Lunar Communications with Libration-Point Satellites," *Journal of Spacecraft and Rockets*, Vol. 4, No. 10, Oct. 1967, pp. 1383–1384.

[3] Farquhar, R. W., "Station-keeping in the Vicinity of Collinear Libration Points with an Application to a Lunar Communications Problem," *Space Flight Mechanics*, Science and Technology Series, American Astronautical Society, New York, 1967, Vol. 11, pp. 519–535.

[4] Porter, J. D., "Final Report for Lunar Libration Point Flight Dynamics Study," NASA GSFC Contract NAS-5-11551, April 1969, General Electric Co., Philadelphia, Pa.

Reply by Author to R. W. Farquhar

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FARQUHAR's comment on the fuel consumption estimates deserves a reply. In his Comment, Eq. (5) properly states the geometrical effects which define A_z; let the value thus computed be denoted A_z^f. Equation (7), which is due to the present author, defines a value which is denoted A_z^h. Then,

$$A_z^f/A_z^h = 1 + \tfrac{1}{2}(\sin \psi_0/k)^2 + 0(1/k^4)$$

Thus, A_z^h may be regarded as a lower bound, which is approximated for moderate values of k. Indeed, $k = 3.0$ gives $(A_z^f/A_z^h) = 1.03$ for $\Delta t = 93$ days. Nevertheless, A_z^f should indeed be used, and I thank Dr. Farquhar for his comment.

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Errata: "Effects of Products of Inertia on Re-Entry Vehicle Roll Behavior"

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IN the above paper, 1) Eq. (6) should read

$$\dot{p} = (1/I_X)\{M_X + (1/I)[J_{XY}(M_Y - I_X pr) + J_{XZ}(M_Z + I_X pq)]\}$$

2) in the section labeled "Conclusion," 2d should read "are zero at zero roll rate"; and 3) in the nomenclature, the fifth symbol defined should be C_{m_q} not C_m.

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